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Variational analysis of transverse cracking and local delamination in $[\theta_m/90_n]_s$ laminates

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Abstract

A displacement-based variational model is developed to study the effects of transverse cracking and local delaminations in symmetric composite laminates. In the model, the crack shape is assumed to be a function of crack density and delamination length. Using a variational approach with the principle of minimum potential energy, governing equations are derived. The effective Young's modulus E_x and energy release rate G are theoretically examined as a result of local delaminations.

Keywords: Variational approach; Transverse cracking; Delamination; Residual stiffness; Young's modulus; Energy release rate

1. Introduction

Laminated composites are widely used in many advanced structural applications. During their usage, composite structures are susceptible to progressive damage. The formation of matrix cracks in 90° layers is the early stage of damage when $[\theta_m/90_n]_s$ composite laminates are subjected to quasi-static or fatigue tensile loading. Experimental data shows that transverse cracking causes stiffness reduction in composite structures. Researchers have presented different approaches to study the damage effect induced by transverse ply cracking on material properties of cracked $[\theta_m/90_n]_s$ laminates.

After transverse matrix cracking in $[\theta_m/90_n]_s$ composite laminates, there is a high interlaminar stress concentration at the crack tip. This stress often induces local delaminations at the interface between laminae. With initiation and growth of local delaminations, the strength and stiffness of composite laminates are significantly impaired. For most composite structures, onset of delaminations emanating from transverse cracks is considered sufficient cause to withdraw the structure from service. Thus, it is necessary to investigate the effect of local delaminations that initiate and grow from transverse crack tips.

Compared with the large number of existing works on transverse ply cracking, only a few approaches have been proposed to study local delaminations in $[\theta_m/90_n]_s$ laminates. O'Brien (1985) derived a closed-form

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equation of strain energy release rate (SERR) G by a simple rule of mixtures analysis and classical laminated plate theory. Caslini et al. (1987) used fracture mechanics to conduct a study on damage initiation and accumulation induced by local delaminations. Selvarathinam and Weitsman (1998, 1999) employed the shear-lag method to model transverse cracking and local delaminations in cross-ply composite laminates under fatigue loading. Berthelot and Le Corre (2000) also used a shear-lag model to evaluate stress distributions in cross-ply composite laminates containing matrix cracks and local delaminations. Thornburgh and Chattopadhyay (2001) used higher-order plate theory to model transverse matrix cracking and local delaminations. Akshantala and Talreja (1998) applied a stress-based variational model to study the evolution of damage induced by matrix cracking and local delaminations in a cross-ply composite laminate. Rebiere and Gamby (2004) modeled the damage behavior of a cross-ply composite laminate under transverse, longitudinal matrix cracking and delaminations using Hashin's approach (1985).

Nairn and Hu (1992) proposed a stress-based variational model to study the local delaminations induced by matrix microcracks in $[S/90_n]_s$ laminates. Zhang et al. (1994) used an improved 2D shear lag model to predict the strain energy release rate G due to matrix cracking and local delaminations in $[\theta_m/90_n]_s$ composite laminates. Then, Zhang et al. (1999) used a first-order shear deformation laminated plate theory and sublaminate method to predict the G in symmetric composite laminates. Kashtalyan and Soutis (2000) applied Zhang's model (1994) to evaluate the stiffness properties of $[\pm\theta_m/90_n]_s$ composite laminates under local delaminations due to transverse cracks and splits. Then, Kashtalyan and Soutis (2002) derived expressions for strain energy release rate of mode I G_{I}, mode II G_{II} and the total G associated with angle-ply matrix cracking and local delaminations based on the 2D shear lag approach.

Above research work is mostly based on shear lag model and stress-based variational approach. In this paper, a displacement-based variational model is proposed to analyze local delaminations in $[\theta_m/90_n]_s$ (θ is assumed as 0° or $\pm\varphi$°) composite laminates. The governing equation is derived from the principle of minimum potential energy. Advantages of the present method are that (1) the true crack shape function is derived as part of the governing equations, (2) unlike the shear lag model no parameters have to be calibrated, and (3) displacement based approach avoids the complex stress boundary conditions at the crack tips. Since the residual stiffness and energy release rate G are two important indices of evaluating the effect of damage due to local delaminations in composite laminates, the energy release rate G and effective Young's modulus E_x as a function of delamination ratio in $[\theta_m/90_n]_s$ laminates are investigated in this paper.

2. Theoretical model

In the present model, a single ply is defined as a lamina and modeled as an orthotropic sheet. The principal material axes as shown in Fig. 1 are longitudinal to the fiber (denoted as 1), transverse to the fiber direction (denoted as 2), and normal to the lamina surface (denoted as 3). The geometry of a laminate with local delamination and transverse cracks under tensile loading is illustrated in Fig. 2. The transverse cracks are assumed to be evenly spaced and to extend across the entire laminate width, and all delaminations are assumed initiated from transverse crack tips simultaneously. After the start of local delaminations, the damage mode is dominated by interlaminar fracture. The unit cell of a cracked $[\theta_m/90_n]_s$ laminate with delamination between two adjacent cracks shown in Fig. 3 is adopted for predicting the effective elastic modulus and energy release rate

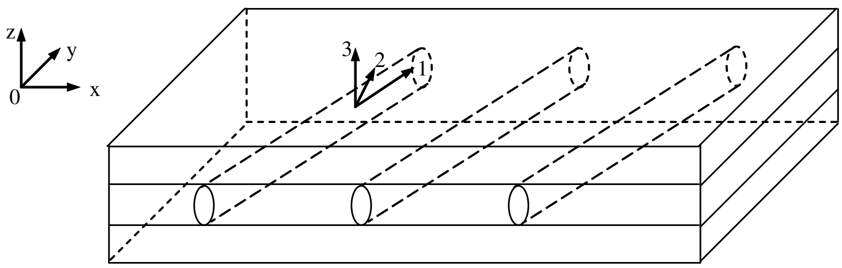

Fig. 1. Definition of the material axes for composite laminate and lamina.

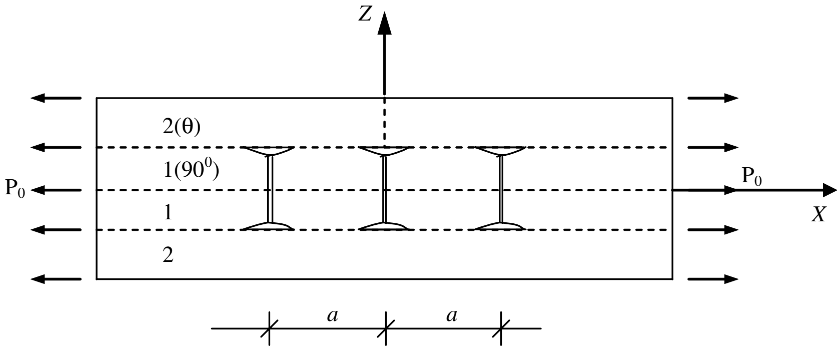

Fig. 2. Local delaminations in a $[\theta/90°]_s$ composite laminate under uniaxial loading.

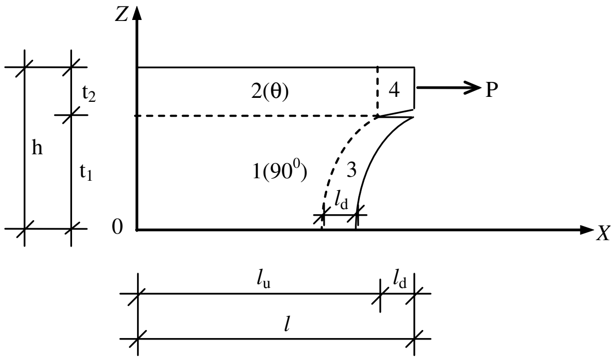

Fig. 3. One-quarter of unit damaged cell with both local delaminations and transverse cracking.

G as a function of delamination ratio. The displacement field is assumed to be independent of the y-coordinate (plane strain).

2.1. Laminated part ($0 \leqslant x \leqslant l_u$)

2.1.1. Sublaminate 1

The displacement functions for sublaminate 1 shown in Fig. 3 are assumed as

$$\begin{aligned} u_x^{(1)}(x,z) &= f_1(x) + \phi(z) f_2(x), \\ u_z^{(1)}(x,z) &= \frac{z^3}{t_1^3} f_3(x), \end{aligned} \tag{1}$$

where $f_1(x)$, $f_2(x)$ and $f_3(x)$ are unknown functions and $\phi(z)$ is the crack shape function.

The boundary conditions for displacement fields are

$$\begin{aligned} u_x^{(1)}(0,z) &= 0, \\ u_x^{(1)}(x,0) &= f_1(x), \\ u_z^{(1)}(x,0) &= 0. \end{aligned} \tag{2}$$

The strain fields are given as

$$\varepsilon_x = \frac{\partial u_x}{\partial x}, \qquad \varepsilon_z = \frac{\partial u_z}{\partial z}, \qquad \gamma_{xz} = \frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}. \tag{3}$$

The constitutive equations are

$$\begin{aligned}
\sigma_x &= c_{xx}\varepsilon_x + c_{xz}\varepsilon_z,\\
\sigma_z &= c_{xz}\varepsilon_x + c_{zz}\varepsilon_z,\\
\tau_{xz} &= c_{55}\gamma_{xz},
\end{aligned} \tag{4}$$

where c_{ij} $(i,j = x, z, 5)$ are elastic stiffness coefficients.

Substituting (1) into (3) and (4), we obtain

$$\begin{aligned}
\varepsilon_x^{(1)} &= f_1'(x) + \phi(z)f_2'(x),\\
\varepsilon_z^{(1)} &= \frac{3z^2}{t_1^3}f_3(x),\\
\gamma_{xz}^{(1)} &= \phi'(z)f_2(x) + \frac{z^3}{t_1^3}f_3'(x),\\
\sigma_x^{(1)} &= c_{xx}^{(1)}\varepsilon_x^{(1)} + c_{xz}^{(1)}\varepsilon_z^{(1)},\\
\sigma_z^{(1)} &= c_{xz}^{(1)}\varepsilon_x^{(1)} + c_{zz}^{(1)}\varepsilon_z^{(1)},\\
\tau_{xz}^{(1)} &= c_{55}^{(1)}\gamma_{xz}^{(1)}.
\end{aligned} \tag{5a–f}$$

From the symmetry conditions of the unit cell with respect to the $x = 0$ and $z = 0$ axes, the following stress conditions are satisfied:

$$\begin{aligned}
\tau_{xz}^{(1)}(x,0) &= 0,\\
\tau_{xz}^{(1)}(0,z) &= 0.
\end{aligned} \tag{6}$$

Applying (1) and (5f) into (2) and (6), we know

$$f_1(0) = 0, \qquad f_2(0) = 0, \qquad f_3'(0) = 0, \qquad \phi(0) = 0, \qquad \phi'(0) = 0. \tag{7}$$

On the other hand, it is assumed that at $x = 0$ the z component of displacement varies only in the z direction, therefore

$$u_z^{(1)}(0,z) = \frac{z^3}{t_1^3}w_1. \tag{8}$$

Then, the boundary value of function $f_3(x)$ is

$$f_3(0) = w_1, \tag{9}$$

where w_1 is an unknown constant.

2.1.2. Sublaminate 2

The boundary and continuity conditions for laminate 2 are

$$\begin{aligned}
u_x^{(2)}(0,z) &= 0,\\
u_x^{(1)}(x,t_1) &= u_x^{(2)}(x,t_1),\\
u_z^{(1)}(x,t_1) &= u_z^{(2)}(x,t_1).
\end{aligned} \tag{10}$$

Then, the displacement fields u_x and u_z are expressed as

$$\begin{aligned}
u_x^{(2)}(x,z) &= f_1(x) + \phi(t_1)f_2(x),\\
u_z^{(2)}(x,z) &= \frac{z}{t_1}\frac{h^2 - z^2}{h^2 - t_1^2}f_3(x) + \frac{z}{h}\frac{t_1^2 - z^2}{t_1^2 - h^2}w_2,
\end{aligned} \tag{11}$$

where w_2 is an unknown constant.

It is obvious that (11) satisfies (10).

Substituting (11) into (3) and (4) again, we obtain

$$\begin{aligned}
&\varepsilon_x^{(2)} = f_1'(x) + \phi(t_1) f_2'(x),\\
&\varepsilon_z^{(2)} = \frac{h^2 - 3z^2}{h^2 - t_1^2}\frac{f_3(x)}{t_1} + \frac{t_1^2 - 3z^2}{t_1^2 - h^2}\frac{w_2}{h},\\
&\gamma_{xz}^{(2)} = \frac{z}{t_1}\frac{h^2 - z^2}{h^2 - t_1^2} f_3'(x),\\
&\sigma_x^{(2)} = c_{xx}^{(2)}\varepsilon_x^{(2)} + c_{xz}^{(2)}\varepsilon_z^{(2)},\\
&\sigma_z^{(2)} = c_{xz}^{(2)}\varepsilon_x^{(2)} + c_{zz}^{(2)}\varepsilon_z^{(2)},\\
&\tau_{xz}^{(2)}(x,z) = c_{55}^{(2)}\gamma_{xz}^{(2)}.
\end{aligned} \tag{12a–f}$$

2.2. Delaminated part ($l_u \leqslant x \leqslant l$)

2.2.1. Sublaminate 3

The stress boundary condition for laminate 3 gives:

$$\tau_{xz}^{(3)}(x,0) = 0. \tag{13}$$

The displacement continuity conditions for laminates 1 and 3 require

$$\begin{aligned}
&u_x^{(1)}(l_u,z) = u_x^{(3)}(l_u,z),\\
&u_z^{(1)}(l_u,z) = u_z^{(3)}(l_u,z).
\end{aligned} \tag{14}$$

Thus, we assume displacement fields of laminate 3 as

$$\begin{aligned}
&u_x^{(3)}(x,z) = \frac{x}{l_u}u_x^{(1)}(l_u,z) + \phi(z)\frac{x}{l_u}\sum_{i=1}^{3} w_{i+2}\left(\frac{x^2 - l_u^2}{l_u^2}\right)^i,\\
&u_z^{(3)}(x,z) = u_z^{(1)}(l_u,z) + \frac{z^3}{t_1^3}\frac{x^2}{l_u^2}\sum_{i=1}^{3} w_{i+5}\left(\frac{x^2 - l_u^2}{l_u^2}\right)^i,
\end{aligned} \tag{15}$$

where w_i ($i = 3,4,5,6,7,8$) are unknown constants.

Substituting (15) into (3) and (4), we have

$$\begin{aligned}
&\varepsilon_x^{(3)} = \frac{f_1(l_u)}{l_u} + \left[\frac{f_2(l_u)}{l_u} + g_0(x)\right]\phi(z),\\
&\varepsilon_z^{(3)} = \frac{3z^2}{t_1^3}\left[f_3(l_u) + \frac{x^2}{l_u^2}\sum_{i=1}^{3} w_{i+5}\left(\frac{x^2 - l_u^2}{l_u^2}\right)^i\right],\\
&\gamma_{xz}^{(3)} = \left[\frac{x}{l_u}f_2(l_u) + \frac{x}{l_u}\sum_{i=1}^{3} w_{i+2}\left(\frac{x^2 - l_u^2}{l_u^2}\right)^i\right]\phi'(z) + \frac{z^3}{t_1^3}g_1(x),\\
&\sigma_x^{(3)} = c_{xx}^{(1)}\varepsilon_x^{(3)} + c_{xz}^{(1)}\varepsilon_z^{(3)},\\
&\sigma_z^{(3)} = c_{xz}^{(1)}\varepsilon_x^{(3)} + c_{zz}^{(1)}\varepsilon_z^{(3)},\\
&\tau_{xz}^{(3)} = c_{55}^{(1)}\gamma_{xz}^{(3)},
\end{aligned} \tag{16a–f}$$

where

$$g_0(x) = \frac{3x^2 - l_u^2}{l_u^3} w_3 + \frac{(5x^2 - l_u^2)(x^2 - l_u^2)}{l_u^5} w_4 + \frac{(7x^2 - l_u^2)(x^2 - l_u^2)^2}{l_u^7} w_5,$$
$$g_1(x) = \frac{2x(2x^2 - l_u^2)}{l_u^4} w_6 + \frac{2x(3x^2 - l_u^2)(x^2 - l_u^2)}{l_u^6} w_7 + \frac{2x(4x^2 - l_u^2)(x^2 - l_u^2)^2}{l_u^8} w_8. \tag{16g}$$

2.2.2. Sublaminate 4

The displacement continuity and boundary conditions for laminate 4 require

$$u_x^{(4)}(l_u, z) = u_x^{(2)}(l_u, z),$$
$$u_z^{(4)}(l_u, z) = u_z^{(2)}(l_u, z), \tag{17}$$
$$\bar{u}_x^{(4)}(l) = u_x^{(4)}(l, t_1),$$

where

$$\bar{u}_x^{(4)}(l) = \frac{1}{t_2} \int_{t_1}^{h} u_x^{(4)}(l, z)\, \mathrm{d}z. \tag{17a}$$

Thus, the displacement fields for laminate 4 are assumed as

$$u_x^{(4)}(x, z) = \frac{x}{l_u} u_x^{(2)}(l_u, z) + \psi_0(z) \frac{x}{l_u} \sum_{i=1}^{3} w_{i+8} \left(\frac{x^2 - l_u^2}{l_u^2} \right)^i,$$
$$u_z^{(4)}(x, z) = u_z^{(2)}(l_u, z) + \psi_1(z) \frac{x^2}{l_u^2} \sum_{i=1}^{3} w_{i+11} \left(\frac{x^2 - l_u^2}{l_u^2} \right)^i, \tag{18}$$

where

$$\psi_0(z) = l_1 z^2 - l_2 z^4 + l_3 z^6,$$
$$\psi_1(z) = \frac{1}{2t_2} \left(\frac{z^3}{h^2} - z \right) \tag{18a}$$

in which l_i ($i = 1, 2, 3$) are listed in appendix (Eqs. (A.1)) and w_i ($i = 9, 10, 11, 12, 13, 14$) are unknown constants.

Substituting (18) into (3) and (4), we have

$$\varepsilon_x^{(4)} = \frac{f_1(l_u)}{l_u} + \phi(t_1) \frac{f_2(l_u)}{l_u} + \psi_0(z) g_2(x),$$

$$\varepsilon_z^{(4)} = \frac{h^2 - 3z^2}{h^2 - t_1^2} \frac{f_3(l_u)}{t_1} + \frac{t_1^2 - 3z^2}{t_1^2 - h^2} \frac{w_2}{h} + \left(\frac{3z^2}{h^2} - 1 \right) \frac{x^2}{2t_2 l_u^2} \sum_{i=1}^{3} w_{i+11} \left(\frac{x^2 - l_u^2}{l_u^2} \right)^i,$$

$$\gamma_{xz}^{(4)} = \psi_2(z) \frac{x}{l_u} \sum_{i=1}^{3} w_{i+8} \left(\frac{x^2 - l_u^2}{l_u^2} \right)^i + \psi_1(z) g_3(x), \tag{19a–f}$$

$$\sigma_x^{(4)} = c_{xx}^{(2)} \varepsilon_x^{(4)} + c_{xz}^{(2)} \varepsilon_z^{(4)},$$

$$\sigma_z^{(4)} = c_{xz}^{(2)} \varepsilon_x^{(4)} + c_{zz}^{(2)} \varepsilon_z^{(4)},$$

$$\tau_{xz}^{(4)} = c_{55}^{(2)} \gamma_{xz}^{(4)},$$

where

$$g_2(x)=\frac{3x^2-l_u^2}{l_u^3}w_9+\frac{(5x^2-l_u^2)(x^2-l_u^2)}{l_u^5}w_{10}+\frac{(7x^2-l_u^2)(x^2-l_u^2)^2}{l_u^7}w_{11},$$
$$g_3(x)=\frac{2x(2x^2-l_u^2)}{l_u^4}w_{12}+\frac{2x(3x^2-l_u^2)(x^2-l_u^2)}{l_u^6}w_{13}+\frac{2x(4x^2-l_u^2)(x^2-l_u^2)^2}{l_u^8}w_{14}, \tag{19g}$$
$$\psi_2(z)=2l_1z-4l_2z^3+6l_3z^5.$$

2.3. Potential energy

The potential energy per unit damaged cell with a unit width is written as

$$\begin{aligned}U=&\frac{1}{2}\int_0^{l_u}\left\{\int_0^{t_1}\left[\sigma_x^{(1)}\varepsilon_x^{(1)}+\sigma_z^{(1)}\varepsilon_z^{(1)}+\tau_{xz}^{(1)}\gamma_{xz}^{(1)}\right]\mathrm{d}z+\int_{t_1}^{h}\left[\sigma_x^{(2)}\varepsilon_x^{(2)}+\sigma_z^{(2)}\varepsilon_z^{(2)}+\tau_{xz}^{(2)}\gamma_{xz}^{(2)}\right]\mathrm{d}z\right\}\mathrm{d}x\\&+\frac{1}{2}\int_{l_u}^{l}\left\{\int_0^{t_1}\left[\sigma_x^{(3)}\varepsilon_x^{(3)}+\sigma_z^{(3)}\varepsilon_z^{(3)}+\tau_{xz}^{(3)}\gamma_{xz}^{(3)}\right]\mathrm{d}z+\int_{t_1}^{h}\left[\sigma_x^{(4)}\varepsilon_x^{(4)}+\sigma_z^{(4)}\varepsilon_z^{(4)}+\tau_{xz}^{(4)}\gamma_{xz}^{(4)}\right]\mathrm{d}z\right\}\mathrm{d}x\\&-P\cdot\bar{u}_x^{(4)}(l),\end{aligned}\tag{20}$$

where

$$P=p_0h \tag{20a}$$

in which p_0 is the applied stress.

The variation of energy U is given as

$$\begin{aligned}\delta U=&\int_0^{l_u}\mathrm{d}x\int_0^{t_1}\left\{c_{xx}^{(1)}\varepsilon_x^{(1)}\delta\varepsilon_x^{(1)}+c_{xz}^{(1)}\left[\varepsilon_z^{(1)}\delta\varepsilon_x^{(1)}+\varepsilon_x^{(1)}\delta\varepsilon_z^{(1)}\right]+c_{zz}^{(1)}\varepsilon_z^{(1)}\delta\varepsilon_z^{(1)}+c_{55}^{(1)}\gamma_{xz}^{(1)}\delta\gamma_{xz}^{(1)}\right\}\mathrm{d}z\\&+\int_0^{l_u}\mathrm{d}x\int_{t_1}^{h}\left\{c_{xx}^{(2)}\varepsilon_x^{(2)}\delta\varepsilon_x^{(2)}+c_{xz}^{(2)}\left[\varepsilon_z^{(2)}\delta\varepsilon_x^{(2)}+\varepsilon_x^{(2)}\delta\varepsilon_z^{(2)}\right]+c_{zz}^{(2)}\varepsilon_z^{(2)}\delta\varepsilon_z^{(2)}+c_{55}^{(2)}\gamma_{xz}^{(2)}\delta\gamma_{xz}^{(2)}\right\}\mathrm{d}z\\&+\int_{l_u}^{l}\mathrm{d}x\int_0^{t_1}\left\{c_{xx}^{(1)}\varepsilon_x^{(3)}\delta\varepsilon_x^{(3)}+c_{xz}^{(1)}\left[\varepsilon_z^{(3)}\delta\varepsilon_x^{(3)}+\varepsilon_x^{(3)}\delta\varepsilon_z^{(3)}\right]+c_{zz}^{(1)}\varepsilon_z^{(3)}\delta\varepsilon_z^{(3)}+c_{55}^{(1)}\gamma_{xz}^{(3)}\delta\gamma_{xz}^{(3)}\right\}\mathrm{d}z\\&+\int_{l_u}^{l}\mathrm{d}x\int_{t_1}^{h}\left\{c_{xx}^{(2)}\varepsilon_x^{(4)}\delta\varepsilon_x^{(4)}+c_{xz}^{(2)}\left[\varepsilon_x^{(4)}\delta\varepsilon_z^{(4)}+\varepsilon_z^{(4)}\delta\varepsilon_x^{(4)}\right]+c_{zz}^{(2)}\varepsilon_z^{(4)}\delta\varepsilon_z^{(4)}+c_{55}^{(2)}\gamma_{xz}^{(4)}\delta\gamma_{xz}^{(4)}\right\}\mathrm{d}z-P\delta\bar{u}_x^{(4)}(l).\end{aligned}\tag{21}$$

From the theorem of minimum potential energy, we have

$$\delta U=0. \tag{22}$$

Therefore, after substituting (5a–c), (12a–c), (16a–c), (19a–c) into (21) with the consideration of (22), for all admissible displacements, there are two governing equations

$$\begin{aligned}&a_1f_1''(x)+a_2f_2''(x)+a_3f_3'(x)=0,\\&a_2f_1''(x)-m_6f_2(x)+a_4f_2''(x)-a_5f_3'(x)=0,\\&a_3f_1'(x)-a_5f_2'(x)+a_6f_3(x)-a_7f_3''(x)-m_{14}w_2=0\end{aligned}\tag{23a}$$

and

$$b_0\phi''(z)-b_1\phi(z)-b_2z^2-b_3=0 \tag{23b}$$

with boundary conditions

$$
\begin{aligned}
&a_1f_1'(l_u)+a_8f_1(l_u)+a_2f_2'(l_u)+a_9f_2(l_u)+a_{10}f_3(l_u)+a_{11}w_2+a_{12}w_3+a_{13}w_4+a_{14}w_5\\
&\quad+a_{15}w_6+a_{16}w_7+a_{17}w_8+a_{18}w_9+a_{19}w_{10}+a_{20}w_{11}+a_{21}w_{12}+a_{22}w_{13}+a_{23}w_{14}-m_{69}P=0,\\
&a_{104}f_1'(l_u)+a_{105}f_1(l_u)+a_{106}f_2'(l_u)+a_{107}f_2(l_u)+a_{108}f_3(l_u)+a_{109}w_2+a_{110}w_3+a_{111}w_4+a_{112}w_5\\
&\quad+a_{113}w_6+a_{114}w_7+a_{115}w_8+a_{116}w_9+a_{117}w_{10}+a_{118}w_{11}+a_{119}w_{12}+a_{120}w_{13}+a_{121}w_{14}=0,\\
&a_{39}f_1(l_u)+a_{40}f_2(l_u)+a_7f_3'(l_u)+a_{41}f_3(l_u)+a_{42}w_2+a_{43}w_3+a_{44}w_4+a_{45}w_5+a_{46}w_6\\
&\quad+a_{47}w_7+a_{48}w_8+a_{49}w_9+a_{50}w_{10}+a_{51}w_{11}+a_{52}w_{12}+a_{53}w_{13}+a_{54}w_{14}=0,\\
&a_{11}f_1(l_u)+a_{26}f_2(l_u)-m_{14}\int_0^{l_u}f_3(x)\,\mathrm{d}x+a_{42}f_3(l_u)+a_{55}w_2+a_{56}w_9+a_{57}w_{10}+a_{58}w_{11}\\
&\quad+a_{59}w_{12}+a_{60}w_{13}+a_{61}w_{14}=0,\\
&a_{12}f_1(l_u)+a_{27}f_2(l_u)+a_{43}f_3(l_u)+a_{62}w_3+a_{63}w_4+a_{64}w_5+a_{65}w_6+a_{66}w_7+a_{67}w_8=0,\\
&a_{13}f_1(l_u)+a_{28}f_2(l_u)+a_{44}f_3(l_u)+a_{63}w_3+a_{68}w_4+a_{69}w_5+a_{70}w_6+a_{71}w_7+a_{72}w_8=0,\\
&a_{14}f_1(l_u)+a_{29}f_2(l_u)+a_{45}f_3(l_u)+a_{64}w_3+a_{69}w_4+a_{73}w_5+a_{74}w_6+a_{75}w_7+a_{76}w_8=0,\\
&a_{15}f_1(l_u)+a_{30}f_2(l_u)+a_{46}f_3(l_u)+a_{65}w_3+a_{70}w_4+a_{74}w_5+a_{77}w_6+a_{78}w_7+a_{79}w_8=0,\\
&a_{16}f_1(l_u)+a_{31}f_2(l_u)+a_{47}f_3(l_u)+a_{66}w_3+a_{71}w_4+a_{75}w_5+a_{78}w_6+a_{80}w_7+a_{81}w_8=0,\\
&a_{17}f_1(l_u)+a_{32}f_2(l_u)+a_{48}f_3(l_u)+a_{67}w_3+a_{72}w_4+a_{76}w_5+a_{79}w_6+a_{81}w_7+a_{82}w_8=0,\\
&a_{18}f_1(l_u)+a_{33}f_2(l_u)+a_{49}f_3(l_u)+a_{56}w_2+a_{83}w_9+a_{84}w_{10}+a_{85}w_{11}+a_{86}w_{12}+a_{87}w_{13}\\
&\quad+a_{88}w_{14}-m_{70}P=0,\\
&a_{19}f_1(l_u)+a_{34}f_2(l_u)+a_{50}f_3(l_u)+a_{57}w_2+a_{84}w_9+a_{89}w_{10}+a_{90}w_{11}+a_{91}w_{12}+a_{92}w_{13}\\
&\quad+a_{93}w_{14}-m_{71}P=0,\\
&a_{20}f_1(l_u)+a_{35}f_2(l_u)+a_{51}f_3(l_u)+a_{58}w_2+a_{85}w_9+a_{90}w_{10}+a_{94}w_{11}+a_{95}w_{12}+a_{96}w_{13}\\
&\quad+a_{97}w_{14}-m_{72}P=0,\\
&a_{21}f_1(l_u)+a_{36}f_2(l_u)+a_{52}f_3(l_u)+a_{59}w_2+a_{86}w_9+a_{91}w_{10}+a_{95}w_{11}+a_{98}w_{12}+a_{99}w_{13}+a_{100}w_{14}=0,\\
&a_{22}f_1(l_u)+a_{37}f_2(l_u)+a_{53}f_3(l_u)+a_{60}w_2+a_{87}w_9+a_{92}w_{10}+a_{96}w_{11}+a_{99}w_{12}+a_{101}w_{13}+a_{102}w_{14}=0,\\
&a_{23}f_1(l_u)+a_{38}f_2(l_u)+a_{54}f_3(l_u)+a_{61}w_2+a_{88}w_9+a_{93}w_{10}+a_{97}w_{11}+a_{100}w_{12}+a_{102}w_{13}+a_{103}w_{14}=0,
\end{aligned}
\tag{23c}
$$

where m_i $(i = 6, 14, 69, 70, 71, 72)$, a_i $(i = 0, 121)$ and b_i $(i = 0, 3)$ are listed in the appendix (Eqs. (A.2)–(A.4)).

Solving (23a) with consideration of (7) and (9), we have the first solution

$$
\begin{aligned}
f_1(x)&=\psi_0x+e_0d_1\sinh(\alpha x)+e_1d_2\sinh(\beta x),\\
f_2(x)&=d_1\sinh(\alpha x)+d_2\sinh(\beta x),\\
f_3(x)&=e_2d_1[1-\cosh(\alpha x)]+e_3d_2[1-\cosh(\beta x)]+w_1
\end{aligned}
\tag{24}
$$

for

$$
\frac{k_1}{k_0}\prec 0,\qquad \frac{k_1^2}{4k_0^2}-\frac{k_2}{k_0}\succ 0,\qquad \frac{k_1}{2k_0}+\sqrt{\frac{k_1^2}{4k_0^2}-\frac{k_2}{k_0}}\prec 0 \tag{24a}
$$

and

$$
\alpha=\sqrt{-\frac{k_1}{2k_0}+\sqrt{\frac{k_1^2}{4k_0^2}-\frac{k_2}{k_0}}},\qquad \beta=\sqrt{-\frac{k_1}{2k_0}-\sqrt{\frac{k_1^2}{4k_0^2}-\frac{k_2}{k_0}}}, \tag{24b}
$$

where k_i $(i = 0, 5)$ are listed in the appendix (Eqs. (A.5)), and

$$\psi_0 = -\frac{a_6}{a_3}\left[e_2 d_1 + e_3 d_2 + w_1 - \frac{m_{14}}{a_6} w_2\right], \qquad e_0 = \frac{1}{a_3}\left[\frac{k_4 + k_5\alpha_0^2}{k_3}\left(\frac{a_6}{\alpha_0^2} - a_7\right) + a_5\right],$$
$$e_1 = \frac{1}{a_3}\left[\frac{k_4 + k_5\beta_0^2}{k_3}\left(\frac{a_6}{\beta_0^2} - a_7\right) + a_5\right], \qquad e_2 = \frac{k_4 + k_5\alpha_0^2}{k_3\alpha_0}, \qquad e_3 = \frac{k_4 + k_5\beta_0^2}{k_3\beta_0}. \tag{24c}$$

The second solution is

$$\begin{aligned} f_1(x) &= \psi_1 x + e_0 d_1 \sinh(\alpha x) - e_4 d_2 \sin(\beta x),\\ f_2(x) &= d_1 \sinh(\alpha x) + d_2 \sin(\beta x),\\ f_3(x) &= e_2 d_1[1 - \cosh(\alpha x)] - e_5 d_2[1 - \cos(\beta x)] + w_1 \end{aligned} \tag{25}$$

for

$$\begin{aligned} &\frac{k_1}{k_0} \prec 0, \qquad \frac{k_1^2}{4k_0^2} - \frac{k_2}{k_0} \succ 0, \qquad \frac{k_1}{2k_0} + \sqrt{\frac{k_1^2}{4k_0^2} - \frac{k_2}{k_0}} \succ 0 \quad \text{or}\\ &\frac{k_1}{k_0} \succ 0, \qquad \frac{k_1^2}{4k_0^2} - \frac{k_2}{k_0} \succ 0, \qquad -\frac{k_1}{2k_0} + \sqrt{\frac{k_1^2}{4k_0^2} - \frac{k_2}{k_0}} \succ 0 \end{aligned} \tag{25a}$$

and

$$\alpha = \sqrt{-\frac{k_1}{2k_0} + \sqrt{\frac{k_1^2}{4k_0^2} - \frac{k_2}{k_0}}}, \qquad \beta = \sqrt{\frac{k_1}{2k_0} + \sqrt{\frac{k_1^2}{4k_0^2} - \frac{k_2}{k_0}}}, \tag{25b}$$

$$\psi_1 = -\frac{a_6}{a_3}\left[e_2 d_1 - e_5 d_2 + w_1 - \frac{m_{14}}{a_6} w_2\right], \qquad e_4 = \frac{1}{a_3}\left[\frac{k_4 - k_5\beta_0^2}{k_3}\left(\frac{a_6}{\beta_0^2} + a_7\right) - a_5\right],$$
$$e_5 = \frac{k_4 - k_5\beta_0^2}{k_3\beta_0}. \tag{25c}$$

The third solution is

$$\begin{aligned} f_1(x) &= \psi_2 x - e_6 d_1 \sin(\alpha x) - e_4 d_2 \sin(\beta x),\\ f_2(x) &= d_1 \sin(\alpha x) + d_2 \sin(\beta x),\\ f_3(x) &= e_7 d_1[\cos(\alpha x) - 1] + e_5 d_2[\cos(\beta x) - 1] + w_1 \end{aligned} \tag{26}$$

for

$$\frac{k_1}{k_0} \succ 0, \qquad \frac{k_1^2}{4k_0^2} - \frac{k_2}{k_0} \succ 0, \qquad \sqrt{\frac{k_1^2}{4k_0^2} - \frac{k_2}{k_0}} \prec \frac{k_1}{2k_0} \tag{26a}$$

and

$$\alpha = \sqrt{\frac{k_1}{2k_0} - \sqrt{\frac{k_1^2}{4k_0^2} - \frac{k_2}{k_0}}}, \qquad \beta = \sqrt{\frac{k_1}{2k_0} + \sqrt{\frac{k_1^2}{4k_0^2} - \frac{k_2}{k_0}}}, \tag{26b}$$

$$\psi_2 = \frac{a_6}{a_3}\left[e_7 d_1 + e_5 d_2 - w_1 + \frac{m_{14}}{a_6} w_2\right], \qquad e_6 = \frac{1}{a_3}\left[\frac{k_4 - k_5\alpha_0^2}{k_3}\left(\frac{a_6}{\alpha_0^2} + a_7\right) - a_5\right],$$
$$e_7 = \frac{k_4 - k_5\alpha_0^2}{k_3\alpha_0}. \tag{26c}$$

The last solution is

$$\begin{aligned} f_1(x) &= \psi_3 x + (k_6 d_1 + k_7 d_2)\sinh(\alpha x)\cos(\beta x) - (k_7 d_1 - k_6 d_2)\cosh(\alpha x)\sin(\beta x), \\ f_2(x) &= d_1 \sinh(\alpha x)\cos(\beta x) + d_2 \cosh(\alpha x)\sin(\beta x), \\ f_3(x) &= (k_8 d_1 + k_9 d_2)\cdot[1 - \cosh(\alpha x)\cos(\beta x)] + (k_9 d_1 - k_8 d_2)\sinh(\alpha x)\sin(\beta x) + w_1 \end{aligned} \tag{27}$$

for

$$\frac{k_1^2}{4k_0^2} - \frac{k_2}{k_0} \prec 0 \tag{27a}$$

and

$$\alpha = \sqrt[4]{\frac{k_2}{k_0}}\cos\left(\frac{\theta}{2}\right), \qquad \beta = \sqrt[4]{\frac{k_2}{k_0}}\sin\left(\frac{\theta}{2}\right), \qquad \tan(\theta) = \sqrt{\frac{4k_0k_2}{k_1^2} - 1}, \tag{27b}$$

where k_i $(i = 6,9)$ are listed in the appendix (Eqs. (A.5)), and

$$\psi_3 = -\frac{a_6}{a_3}\left[k_8 d_1 + k_9 d_2 + w_1 - \frac{m_{14}}{a_6} w_2\right]. \tag{27c}$$

Then, solving (23b), we obtain

$$\phi(z) = c_0 \sinh(\lambda_1 z) + c_1 \cosh(\lambda_1 z) - c_2 z^2 - c_3, \tag{28}$$

where

$$\lambda_1 = \sqrt{\frac{b_1}{b_0}} \succ 0, \qquad c_2 = \frac{b_2}{b_1}, \qquad c_3 = \frac{2b_0b_2 + b_1b_3}{b_1^2}. \tag{28a}$$

From (7), we have

$$c_0 = 0, \qquad c_1 = c_3. \tag{29}$$

Thus,

$$\phi(z) = c_3[\cosh(\lambda_1 z) - \lambda_2 z^2 - 1], \tag{30}$$

where

$$\lambda_2 = \frac{c_2}{c_3}. \tag{30a}$$

On the other hand, the expressions of $f_1(x)$, $f_2(x)$ and $f_3(x)$ have already considered the constant c_3, therefore $\phi(z)$ is

$$\phi(z) = \cosh(\lambda_1 z) - \lambda_2 z^2 - 1. \tag{31}$$

3. Effective modulus

The effective elastic modulus in $[\theta_m/90_n]_s$ laminates with transverse cracking and local delaminations generally depends on the crack density and delamination ratio. The crack density ψ and delamination ratio ρ are defined as

$$\psi = \frac{t_1}{l}, \qquad \rho = \frac{l_d}{l}, \tag{32}$$

where t_1 is the thickness of the 90° ply, l_d is the delamination length and l is the half length of unit cell. When we know the crack density ψ and delamination ratio ρ, we can solve λ_1, λ_2, α, β, d_1, d_2 and w_i $(i = 1,14)$ from (24b), (25b), (26b), (27b), (28a), (30a) and (23c) with a numerical method. On the other hand, Young's modulus E_0 for an uncracked laminate is determined by lamination theory. After matrix cracking and local delaminations, the effective longitudinal Young's modulus E_x is calculated as

$$E_x = \frac{p_0}{\bar{\varepsilon}_x}, \tag{33}$$

where

$$\bar{\varepsilon}_x = \frac{1}{t_2 l}\int_0^{l_u} \mathrm{d}x \int_{t_1}^{h} \varepsilon_x^{(2)}\, \mathrm{d}z + \frac{1}{t_2 l}\int_{l_u}^{l_u} \mathrm{d}x \int_{t_1}^{h} \varepsilon_x^{(4)}\, \mathrm{d}z. \tag{33a}$$

Introducing (12a) and (19a) into (33) with consideration of (33a) gives

$$E_x = \frac{p_0 l_u}{f_1(l) + \phi(t_1) f_2(l) + \sum_{i=1}^{3}\left(\frac{l^2}{l_u^2} - 1\right)^i w_{i+8}}. \tag{34}$$

4. Energy release rate

The energy release rate is an important index for evaluating local delaminations induced by transverse matrix cracking. The energy release rate G is defined as

$$G = -\frac{\mathrm{d}U_{\mathrm{Total}}}{\mathrm{d}A}, \tag{35}$$

where U_{Total} is the total potential energy of the laminate with a unit width and A is the total delamination area. If a $[\theta_m/90_n]_s$ composite laminate has N microcrack intervals associated with a delamination length l_{d}, U_{Total} and A are given as

$$U_{\mathrm{Total}} = 4NU, \qquad A = 8Nl_{\mathrm{d}}, \tag{36}$$

where U is the potential energy per unit cell which is given in (20). Substituting (36) into (35) and using a finite-difference equation for the derivative, the energy release rate G for local delaminations is obtained as

$$G = -\frac{U(l_{\mathrm{d}} + \Delta a) - U(l_{\mathrm{d}})}{2\Delta a} \quad \text{when } \Delta a \to 0. \tag{37}$$

If the computed energy release rate G remains constant with decreasing values of Δa, convergence is guaranteed.

5. Results and discussion

In this paper, analytical investigation of local delaminations in $[\theta_m/90_n]_s$ composite laminates due to transverse cracking is performed. Two composite materials reported by Ogihara and Takeda (1995) and Joffe and Varna (1999) are selected to investigate the effective longitudinal Young's modulus E_x and energy release rate G as a result of local delaminations that emanate from transverse cracks. One is a glass/epoxy composite and the other is a T800H/3631 carbon/epoxy composite. The corresponding properties are listed in Table 1.

The changes in longitudinal Young's modulus E_x as a function of crack density for T800H/3631 $[0/90_n]_s$ $(n = 2, 4, 6)$ laminates when the delamination ratio ρ equals 0.1 are shown in Figs. 4–6. Kashtalyan and Soutis (2005) also gave their predicted results based on shear-lag approach. Compared with experimental results, the present model gives an accurate prediction. It indicates that our model is able to predict the degradation of E_x due to local delaminations for carbon/epoxy cross-ply laminates. Results also show that the stiffness loss

Table 1
Material properties of composites

Material	E_L (GPa)	E_T (GPa)	G_L (GPa)	G_T (GPa)	vv_L	vv_T	h_0 (mm)
Glass/epoxy	44.73	12.76	5.8	4.49	0.297	0.42	0.144
T800H/3631[a]	148	9.57	4.5	4.2[b]	0.356	0.49	0.132

[a] Material property is selected from Mizutanil et al. (2003).
[b] Unknown value is selected from other composite materials.

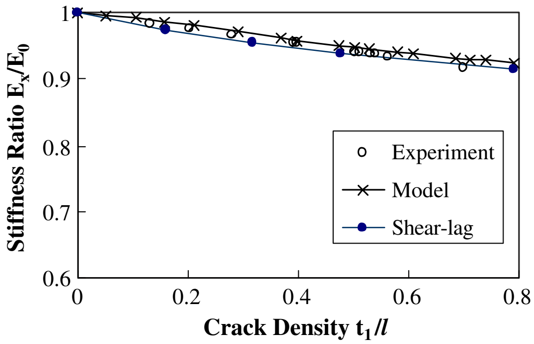

Fig. 4. Effective stiffness E_x as a function of crack density for T800H/3631 $[0/90_2]_s$ laminates for delamination ratio $\rho = 0.1$.

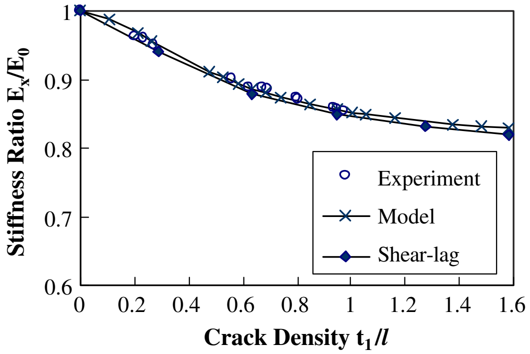

Fig. 5. Effective stiffness E_x as a function of crack density for T800H/3631 $[0/90_4]_s$ laminates for delamination ratio $\rho = 0.1$.

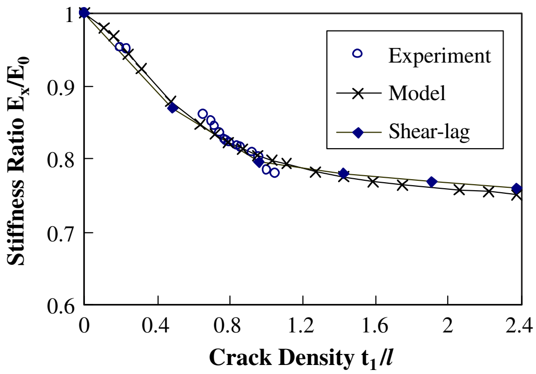

Fig. 6. Effective stiffness E_x as a function of crack density for T800H/3631 $[0/90_6]_s$ laminates for delamination ratio $\rho = 0.1$.

in $[0/90_6]_s$ laminates is more than that in $[0/90_2]_s$ and $[0/90_4]_s$ laminates. The reason is that 90° layers take more portion of the applied load than 0° layer because of a greater relative thickness of 90° layers in the uncracked $[0/90_6]_s$ laminates, so that local delaminations lead to more loss in load-carrying capacity for the $[0/90_6]_s$ laminates. Therefore, the thickness ratio h_{90}/h_0 plays an important role in the residual stiffness for damaged composite laminates. Although the shear-lag model also gives good results, its accuracy depends on shear-lag parameters that are calibrated.

The degradation of longitudinal Young's modulus E_x due to local delaminations for $[\pm\varphi/90_4]_s$ glass/epoxy laminates when the crack density $\psi = 0.75$ is plotted in Fig. 7. It shows that the stiffness loss increases with increasing ply orientation angle φ under the same delamination ratio. Local delaminations cause more stiffness

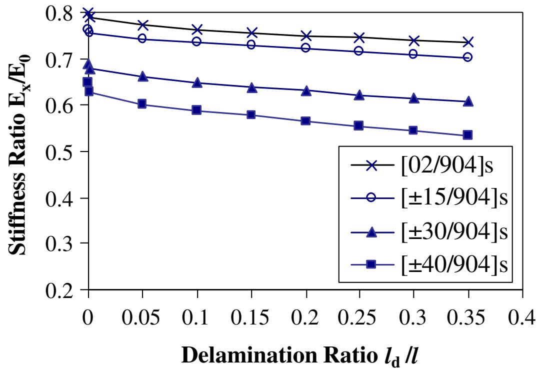

Fig. 7. The relation between delamination length and stiffness ratio for glass/epoxy $[\pm\varphi/90_4]_s$ laminates.

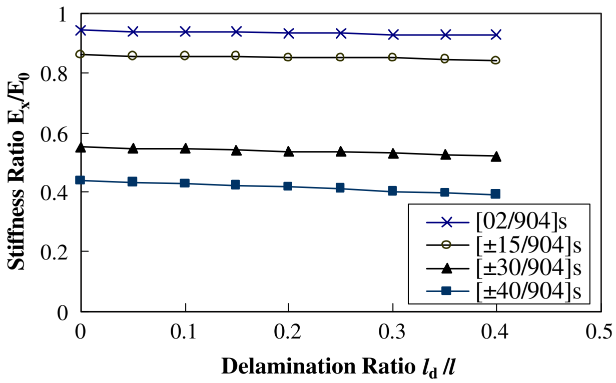

Fig. 8. The relation between delamination length and stiffness ratio for T800H/3631 $[\pm\varphi/90_4]_s$ laminates.

degradation in glass/epoxy laminates when $\varphi > 15°$. It indicates that the ability of the structural damage resistance for $[\pm\varphi/90_4]_s$ glass/epoxy laminates becomes lower with increasing φ.

The changes in longitudinal Young's modulus E_x as a function of delamination length for T800H/3631 $[\pm\varphi/90_4]_s$ laminates when the crack density $\psi = 0.75$ are shown in Fig. 8. It shows that the stiffness loss in $[\pm\varphi/90_4]_s$ laminates is more notable when $\varphi \geqslant 30°$. With increasing delamination length, the stiffness loss due to transverse cracking and local delaminations in the $[\pm 40/90_4]_s$ carbon/epoxy laminate is more than that in other angle-ply laminates. It appears that the degree of the ply angle φ has a significant influence on the residual stiffness of $[\pm\varphi/90_4]_s$ carbon/epoxy laminates if $\varphi \geqslant 30°$.

Fig. 9 shows the relationship between the normalized energy release rate and delamination length for $[\pm\varphi/90_4]_s$ glass/epoxy laminates when the crack density ψ is 0.75. It indicates that the energy release rate increases with increasing ply orientation angle φ, which means local delaminations will propagate more unstably in $[\pm\varphi/90_4]_s$ glass/epoxy laminates with increasing values of φ. On the other hand, Fig. 9 also illustrates that changes in the energy release rate are small when $\varphi \leqslant 15$. It indicates the propagation of local delaminations has a similar pattern in $[\pm\varphi/90_4]_s$ glass/epoxy laminates if the angle φ is sufficiently small.

Fig. 10 shows the relation of the delamination length and normalized energy release rate for $[\pm\varphi/90_4]_s$ carbon/epoxy laminates when the crack density ψ is 0.75. Similar to the case of $[\pm\varphi/90_4]_s$ glass/epoxy laminates, the energy release rate also increases when the angle φ increases. When $\varphi = 40°$, the growth of delamination will be much more unstable. In addition, comparing Figs. 9 and 10, it is obvious that the energy release rate of $[\pm\varphi/90_4]_s$ glass/epoxy laminate increases more than that of $[\pm\varphi/90_4]_s$ carbon/

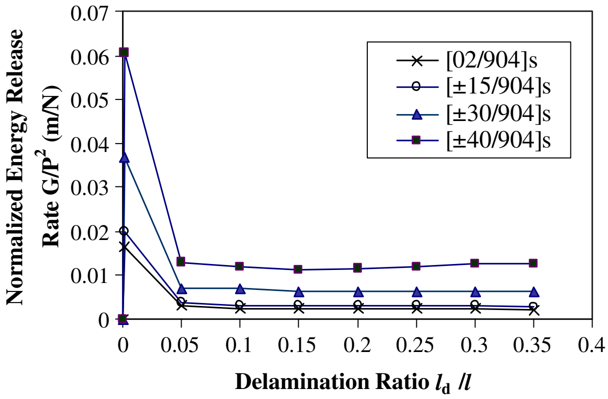

Fig. 9. The relation between delamination length and normalized energy release rate for glass/epoxy $[\pm\varphi/90_4]_s$ laminates.

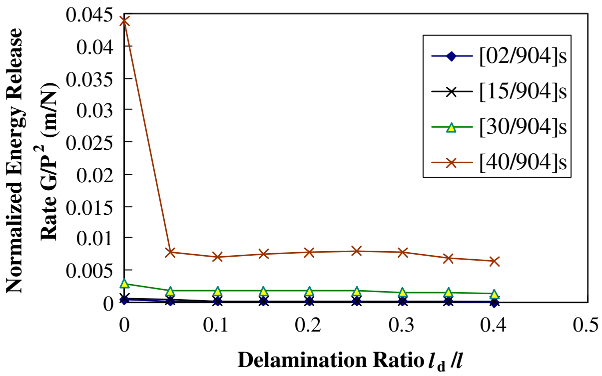

Fig. 10. The relation between delamination length and normalized energy release rate for T800H/3631 $[\pm\varphi/90_4]_s$ laminates.

epoxy laminate when $\varphi \leqslant 40$. Therefore, under the same configuration and crack density, the propagation of local delaminations is more stable in angle-ply carbon/epoxy laminates than in glass/epoxy laminates.

6. Summary and conclusions

In this paper, a displacement-based model is developed to predict the effective longitudinal Young's modulus E_x and energy release rate G in $[\theta_m/90_n]_s$ composite laminates with transverse matrix cracking and local delaminations. The governing equations are derived based on the variational theory and principle of minimum potential energy. In the model, the crack shape as well as E_x and G are functions of crack density and delamination length. The effect of local delaminations on E_x and G is examined by using glass/epoxy and carbon/epoxy $[\pm\varphi/90_4]_s$ composite laminates, where $\varphi = 0°, 15°, 30°, 40°$. From the results, we conclude that:

1. With increasing local delaminations, the predicted changes in E_x and G well reflect damage effects in glass/epoxy and carbon/epoxy $[\theta_m/90_n]_s$ composite laminates. Thus, for different $[\theta_m/90_n]_s$ composite materials, this model can evaluate the damaged structural response at different stages of degradation.
2. The local delaminations induced by transverse matrix cracks have significant influence on E_x and G, especially for glass/epoxy $[\theta_m/90_n]_s$ laminates. Thus, accurate prediction of the initiation and growth of local delaminations is an important research area for characterizing damage propagation in glass/epoxy composite laminates.

3. The damage influence on E_x and G indicates that there is much more damage accumulation in composite laminates with increasing ply orientation angle θ. Thus, accurate assessment of progressive damage and fracture characteristics in the form of transverse cracking and local delaminations is critical for practical applications of $[\theta_m/90_n]_s$ composite laminates.
4. If constitutive equations include thermal and moisture effects, this model can also be used to predict the E_x and G in "hot-wet" conditions.
5. The developed model can be extended to study local delaminations induced by the random distribution of transverse cracks in composite laminates. In addition, this model can be applied to calculate the individual components of the energy release rate for local delaminations.

Appendix

$$\begin{aligned}
l_0 &= 33h^9 - 90t_1^2h^7 + 49t_1^4h^5 + 42t_1^5h^4 - 40t_1^7h^2 + 6t_1^9,\\
l_1 &= \frac{3h^2}{l_0t_1^2}(11h^7 - 21t_1^4h^3 - 84t_1^4t_2h^2 + 10t_1^6h + 60t_1^6t_2),\\
l_2 &= \frac{15}{l_0t_1^2}(6h^7 - 7t_1^2h^5 - 14t_1^2t_2h^4 + 7t_1^6t_2 + t_1^7),\\
l_3 &= \frac{7}{l_0t_1^2}(7h^5 - 10t_1^2h^3 - 20t_1^2t_2h^2 + 15t_1^4t_2 + 3t_1^5).
\end{aligned} \tag{A.1}$$

$$\begin{aligned}
&m_1 = c_{xx}^{(1)}t_1, \qquad m_2 = c_{xx}^{(2)}t_2, \qquad m_3 = c_{xx}^{(1)}\int_0^{t_1}\phi(z)\,\mathrm{d}z, \qquad m_4 = c_{xz}^{(1)}, \qquad m_5 = c_{xz}^{(2)},\\
&m_6 = c_{55}^{(1)}\int_0^{t_1}[\phi'(z)]^2\,\mathrm{d}z, \qquad m_7 = c_{xx}^{(1)}\int_0^{t_1}\phi^2(z)\,\mathrm{d}z, \qquad m_8 = c_{55}^{(1)}\left[\phi(t_1) - \frac{3}{t_1^3}\int_0^{t_1}z^2\phi(z)\,\mathrm{d}z\right],\\
&m_9 = c_{xz}^{(1)}\frac{3}{t_1^3}\int_0^{t_1}z^2\phi(z)\,\mathrm{d}z, \qquad m_{10} = c_{zz}^{(1)}\frac{9}{5t_1}, \qquad m_{11} = c_{zz}^{(2)}\frac{20t_1^3h + 15t_1t_2^2h + 5t_1^2t_2^2 + 4t_2^4}{5t_1^2t_2(h+t_1)^2},\\
&m_{12} = c_{55}^{(1)}\frac{t_1}{7}, \qquad m_{13} = c_{55}^{(2)}t_2\frac{140t_1^3h + 105t_1^2t_2h + 56t_1t_2^2h + 7t_1^2t_2^2 + 8t_2^4}{105t_1^2(h+t_1)^2},\\
&m_{14} = 4c_{zz}^{(2)}\frac{5t_1^4 + 10t_1^2t_2h + 5t_1t_2^3 + t_2^4}{5ht_1t_2(h+t_1)^2}, \qquad m_{15} = c_{zz}^{(2)}\frac{20t_1^3h + 40t_1t_2h^2 + 5t_2^3h + 4t_2^4}{5h^2t_2(h+t_1)^2}l_u,\\
&m_{16} = \frac{l^5}{l_u^6} - \frac{2l^3}{l_u^4} + \frac{l}{l_u^2}, \qquad m_{17} = \frac{l^7}{l_u^8} - 3\frac{l^5}{l_u^6} + 3\frac{l^3}{l_u^4} - \frac{l}{l_u^2}, \qquad m_{18} = \frac{l^5}{5l_u^4} + \frac{2}{15}l_u - \frac{l^3}{3l_u^2},\\
&m_{19} = \frac{l^7}{7l_u^6} - \frac{8}{105}l_u - \frac{2l^5}{5l_u^4} + \frac{l^3}{3l_u^2}, \qquad m_{20} = \frac{l^9}{9l_u^8} + \frac{16}{315}l_u - \frac{3l^7}{7l_u^6} + \frac{3l^5}{5l_u^4} - \frac{l^3}{3l_u^2},\\
&m_{21} = \left(\frac{t_1^9}{3} - \frac{4}{21}h^9 - \frac{h^2}{7}t_1^7\right)\frac{l_3}{t_1(h^2-t_1^2)} + \left(\frac{8}{35}h^7 + \frac{h^2}{5}t_1^5 - \frac{3}{7}t_1^7\right)\frac{l_2}{t_1(h^2-t_1^2)}\\
&\qquad + \left(\frac{3}{5}t_1^5 - \frac{4}{15}h^5 - \frac{h^2}{3}t_1^3\right)\frac{l_1}{t_1(h^2-t_1^2)}, \qquad m_{22} = \frac{20t_1^3h + 15t_1t_2^2h + 5t_1^2t_2^2 + 4t_2^4}{10t_1h^2(h^2-t_1^2)},\\
&m_{23} = \left(\frac{h^9}{3} - \frac{4}{21}t_1^9 - \frac{h^7}{7}t_1^2\right)\frac{l_3}{h(h^2-t_1^2)} + \left(\frac{8}{35}t_1^7 + \frac{h^5}{5}t_1^2 - \frac{3}{7}h^7\right)\frac{l_2}{h(h^2-t_1^2)}\\
&\qquad + \left(\frac{3}{5}h^5 - \frac{4}{15}t_1^5 - \frac{h^3}{3}t_1^2\right)\frac{l_1}{h(h^2-t_1^2)}, \qquad m_{24} = \frac{2}{5}\frac{5t_1^3h + 5t_1t_2h^2 + t_2^4}{h^3(h^2-t_1^2)},
\end{aligned}$$

$$m_{25}=\frac{2l^5}{5l_u^5}-\frac{1}{15}-\frac{l^3}{3l_u^3},\qquad m_{26}=\frac{3l^7}{7l_u^7}+\frac{4}{105}-\frac{4l^5}{5l_u^5}+\frac{l^3}{3l_u^3},\qquad m_{27}=\frac{4l^9}{9l_u^9}-\frac{8}{315}-\frac{9l^7}{7l_u^7}+\frac{6l^5}{5l_u^5}-\frac{l^3}{3l_u^3},$$

$$m_{28}=\frac{l_3^2}{13}(h^{13}-t_1^{13})-\frac{2}{11}(h^{11}-t_1^{11})l_2l_3+\frac{2}{9}(h^9-t_1^9)l_1l_3+\frac{l_2^2}{9}(h^9-t_1^9)-\frac{2}{7}(h^7-t_1^7)l_1l_2$$
$$+\frac{l_1^2}{5}(h^5-t_1^5),\qquad m_{29}=\frac{9l^5}{5l_u^6}-\frac{4}{5l_u}-\frac{2l^3}{l_u^4}+\frac{l}{l_u^2},$$

$$m_{30}=\frac{36}{11}(h^{11}-t_1^{11})l_3^2-\frac{16}{3}(h^9-t_1^9)l_2l_3+\frac{24}{7}(h^7-t_1^7)l_1l_3+\frac{16}{7}(h^7-t_1^7)l_2^2-\frac{16}{5}(h^5-t_1^5)l_1l_2$$
$$+\frac{4}{3}(h^3-t_1^3)l_1^2,\qquad m_{31}=\frac{15l^7}{7l_u^8}+\frac{16}{35l_u}-\frac{23l^5}{5l_u^6}+3\frac{l^3}{l_u^4}-\frac{l}{l_u^2},$$

$$m_{32}=\frac{7l^9}{3l_u^{10}}-\frac{32}{105l_u}-\frac{52l^7}{7l_u^8}+\frac{42l^5}{5l_u^6}-\frac{4l^3}{l_u^4}+\frac{l}{l_u^2},\qquad m_{33}=\frac{l^{11}}{11l_u^{10}}-\frac{128}{3465}l_u-\frac{4l^9}{9l_u^8}+\frac{6l^7}{7l_u^6}-\frac{4l^5}{5l_u^4}+\frac{l^3}{3l_u^2},$$

$$m_{34}=\frac{l_3}{t_2}\left(\frac{2}{21}h^7-\frac{t_1^9}{6h^2}+\frac{t_1^7}{14}\right)-\frac{l_2}{t_2}\left(\frac{4}{35}h^5-\frac{3t_1^7}{14h^2}+\frac{t_1^5}{10}\right)+\frac{l_1}{t_2}\left(\frac{2}{15}h^3-\frac{3t_1^5}{10h^2}+\frac{t_1^3}{6}\right),$$

$$m_{35}=\frac{l_3}{t_2}\left(\frac{3}{7}t_1^7-\frac{2}{21}h^7-\frac{t_1^9}{3h^2}\right)+\frac{l_2}{t_2}\left(\frac{4}{35}h^5+\frac{2t_1^7}{7h^2}-\frac{2}{5}t_1^5\right)-\frac{l_1}{t_2}\left(\frac{2}{15}h^3+\frac{t_1^5}{5h^2}-\frac{t_1^3}{3}\right),$$

$$m_{36}=\frac{2l^7}{7l_u^7}-\frac{2}{105}-\frac{3l^5}{5l_u^5}+\frac{l^3}{3l_u^3},\qquad m_{37}=\frac{l^9}{3l_u^7}-\frac{l^7}{l_u^5}+\frac{l^5}{l_u^3}-\frac{l^3}{3l_u},$$

$$m_{38}=\frac{3l^{11}}{11l_u^9}-\frac{32}{3465}l_u^2-\frac{10l^9}{9l_u^7}+\frac{12l^7}{7l_u^5}-\frac{6l^5}{5l_u^3}+\frac{l^3}{3l_u},$$

$$m_{39}=\frac{4l^{11}}{11l_u^{11}}+\frac{16}{3465}-\frac{13l^9}{9l_u^9}+\frac{15l^7}{7l_u^7}-\frac{7l^5}{5l_u^5}+\frac{l^3}{3l_u^3},\qquad m_{40}=\frac{25l^9}{9l_u^{10}}-\frac{128}{315l_u}-\frac{60l^7}{7l_u^8}+\frac{46l^5}{5l_u^6}-\frac{4l^3}{l_u^4}+\frac{l}{l_u^2},$$

$$m_{41}=\frac{35l^{11}}{11l_u^{12}}+\frac{128}{385l_u}-13\frac{l^9}{l_u^{10}}+\frac{142l^7}{7l_u^8}-\frac{74l^5}{5l_u^6}+5\frac{l^3}{l_u^4}-\frac{l}{l_u^2},$$

$$m_{42}=\frac{l^{13}}{13l_u^{12}}+\frac{256}{9009}l_u-\frac{5l^{11}}{11l_u^{10}}+\frac{10l^9}{9l_u^8}-\frac{10l^7}{7l_u^6}+\frac{l^5}{l_u^4}-\frac{l^3}{3l_u^2},$$

$$m_{43}=\frac{5l^9}{9l_u^7}-\frac{16}{315}l_u^2-\frac{11l^7}{7l_u^5}+\frac{7l^5}{5l_u^3}-\frac{l^3}{3l_u},\qquad m_{44}=\frac{2l^9}{9l_u^9}+\frac{8}{315}-\frac{5l^7}{7l_u^7}+\frac{4l^5}{5l_u^5}-\frac{l^3}{3l_u^3},$$

$$m_{45}=\frac{5l^{11}}{11l_u^{10}}+\frac{64}{3465}l_u-\frac{16l^9}{9l_u^8}+\frac{18l^7}{7l_u^6}-\frac{8l^5}{5l_u^4}+\frac{l^3}{3l_u^2},$$

$$m_{46}=\frac{5l^{13}}{13l_u^{11}}-\frac{256}{45,045}l_u^2-\frac{21l^{11}}{11l_u^9}+\frac{34l^9}{9l_u^7}-\frac{26l^7}{7l_u^5}-\frac{9l^5}{5l_u^3}-\frac{l^3}{3l_u},$$

$$m_{47}=\frac{4l^{13}}{13l_u^{13}}+\frac{128}{45,045}-\frac{17l^{11}}{11l_u^{11}}+\frac{28l^9}{9l_u^9}-\frac{22l^7}{7l_u^7}+\frac{8l^5}{5l_u^5}-\frac{l^3}{3l_u^3},$$

$$m_{48}=\frac{49l^{13}}{13l_u^{14}}-\frac{1536}{5005l_u}-\frac{210l^{11}}{11l_u^{12}}+\frac{39l^9}{l_u^{10}}-\frac{284l^7}{7l_u^8}+\frac{111l^5}{5l_u^6}-6\frac{l^3}{l_u^4}+\frac{l}{l_u^2},$$

$$m_{49}=\frac{l^{15}}{15l_u^{14}}-\frac{1024}{45,045}l_u-\frac{6l^{13}}{13l_u^{12}}+\frac{15l^{11}}{11l_u^{10}}-\frac{20l^9}{9l_u^8}+\frac{15l^7}{7l_u^6}-\frac{6l^5}{5l_u^4}+\frac{l^3}{3l_u^2},$$

$$m_{50} = \frac{7l^{11}}{11l_u^9} + \frac{32}{693}l_u^2 - \frac{22l^9}{9l_u^7} + \frac{24l^7}{7l_u^5} - 2\frac{l^5}{l_u^3} + \frac{l^3}{3l_u}, \qquad m_{51} = \frac{2l^{11}}{11l_u^{11}} - \frac{16}{693} - \frac{7l^9}{9l_u^9} + \frac{9l^7}{7l_u^7} - \frac{l^5}{l_u^5} + \frac{l^3}{3l_u^3},$$
$$m_{52} = \frac{7l^{13}}{13l_u^{11}} - \frac{1024}{45,045}l_u^2 - \frac{29l^{11}}{11l_u^9} + \frac{46l^9}{9l_u^7} - \frac{34l^7}{7l_u^5} + \frac{11l^5}{5l_u^3} - \frac{l^3}{3l_u},$$
$$m_{53} = \frac{3l^{13}}{13l_u^{11}} + \frac{512}{45,045}l_u^2 - \frac{13l^{11}}{11l_u^9} + \frac{22l^9}{9l_u^7} - \frac{18l^7}{7l_u^5} + \frac{7l^5}{5l_u^3} - \frac{l^3}{3l_u},$$
$$m_{54} = \frac{7l^{15}}{15l_u^{14}} + \frac{512}{45,045}l_u - \frac{36l^{13}}{13l_u^{12}} + \frac{751l^{11}}{11l_u^{10}} - \frac{80l^9}{9l_u^8} + \frac{45l^7}{7l_u^6} - \frac{12l^5}{5l_u^4} + \frac{l^3}{3l_u^2},$$
$$m_{55} = \frac{4l^{15}}{15l_u^{15}} - \frac{256}{45,045} - \frac{21l^{13}}{13l_u^{13}} + \frac{45l^{11}}{11l_u^{11}} - \frac{50l^9}{9l_u^9} + \frac{30l^7}{7l_u^7} - \frac{9l^5}{5l_u^5} + \frac{l^3}{3l_u^3},$$
$$m_{56} = \frac{20t_1^4 + 20t_1^3t_2 + 20t_1^2t_2^2 + 15t_1t_2^3 + 4t_2^4}{20h^4}, \qquad m_{57} = \frac{l^9}{9l_u^7} - \frac{8}{315}l_u^2 - \frac{2l^7}{7l_u^5} + \frac{l^5}{5l_u^3},$$
$$m_{58} = \frac{140t_1^4 + 245t_1^3t_2 + 168t_1^2t_2^2 + 56t_1t_2^3 + 8t_2^4}{105h^4}t_2, \qquad m_{59} = \frac{4l^7}{7l_u^8} - \frac{11}{105l_u} - \frac{4l^5}{5l_u^6} + \frac{l^3}{3l_u^4},$$
$$m_{60} = \frac{l^{11}}{11l_u^9} + \frac{16}{1155}l_u^2 - \frac{l^9}{3l_u^7} + \frac{3l^7}{7l_u^5} - \frac{l^5}{5l_u^3}, \qquad m_{61} = \frac{2l^9}{3l_u^{10}} + \frac{4}{105l_u} - \frac{11l^7}{7l_u^8} + \frac{6l^5}{5l_u^6} - \frac{l^3}{3l_u^4}, \frac{l^7}{l_u^2} - \frac{l^5}{5},$$
$$m_{62} = \frac{l^{13}}{13l_u^{11}} - \frac{128}{15015}l_u^2 - \frac{4l^{11}}{11l_u^9} + \frac{2l^9}{3l_u^7} - \frac{4l^7}{7l_u^5} + \frac{l^5}{5l_u^3},$$
$$m_{63} = \frac{8l^{11}}{11l_u^{12}} - \frac{8}{495l_u} - \frac{22l^9}{9l_u^{10}} + 3\frac{l^7}{l_u^8} - \frac{8l^5}{5l_u^6} + \frac{l^3}{3l_u^4},$$
$$m_{64} = \frac{9l^{11}}{11l_u^{12}} - \frac{32}{1155l_u} - \frac{8l^9}{3l_u^{10}} + \frac{22l^7}{7l_u^8} - \frac{8l^5}{5l_u^6} + \frac{l^3}{3l_u^4},$$
$$m_{65} = \frac{l^{15}}{15l_u^{13}} + \frac{256}{45,045}l_u^2 - \frac{5l^{13}}{13l_u^{11}} + \frac{10l^{11}}{11l_u^9} - \frac{10l^9}{9l_u^7} + \frac{5l^7}{7l_u^5} - \frac{l^5}{5l_u^3},$$
$$m_{66} = \frac{12l^{13}}{13l_u^{14}} + \frac{160}{9009l_u} - \frac{43l^{11}}{11l_u^{12}} + \frac{58l^9}{9l_u^{10}} - \frac{36l^7}{7l_u^8} + 2\frac{l^5}{l_u^6} - \frac{l^3}{3l_u^4},$$
$$m_{67} = \frac{l^{17}}{17l_u^{15}} - \frac{1024}{255,255}l_u^2 - \frac{2l^{15}}{5l_u^{13}} + \frac{15l^{13}}{13l_u^{11}} - \frac{20l^{11}}{11l_u^9} + \frac{5l^9}{3l_u^7} - \frac{6l^7}{7l_u^5} + \frac{l^5}{5l_u^3},$$
$$m_{68} = \frac{16l^{15}}{15l_u^{16}} - \frac{128}{9009l_u} - \frac{72l^{13}}{13l_u^{14}} + \frac{129l^{11}}{11l_u^{12}} - \frac{116l^9}{9l_u^{10}} + \frac{54l^7}{7l_u^8} - \frac{12l^5}{5l_u^6} + \frac{l^3}{3l_u^4}, \qquad m_{69} = \frac{l}{l_u},$$
$$m_{70} = \frac{l}{l_u}\left(\frac{l^2}{l_u^2} - 1\right), \qquad m_{71} = \frac{l}{l_u}\left(\frac{l^2}{l_u^2} - 1\right)^2, \qquad m_{72} = \frac{l}{l_u}\left(\frac{l^2}{l_u^2} - 1\right)^3. \tag{A.2}$$

$$a_1 = m_1 + m_2, \qquad a_2 = m_2\phi(t_1) + m_3, \qquad a_3 = m_4 - m_5, \qquad a_4 = m_7 + m_2\phi^2(t_1),$$
$$a_5 = m_8 - m_9 + m_5\phi(t_1), \qquad a_6 = m_{10} + m_{11}, \qquad a_7 = m_{12} + m_{13},$$
$$a_8 = \frac{m_1}{l_u}\left(\frac{l}{l_u} - 1\right) + c_{xx}^{(2)}\frac{t_2}{l_u}\left(\frac{l}{l_u} - 1\right), \qquad a_9 = \frac{m_3}{l_u}\left(\frac{l}{l_u} - 1\right) + c_{xx}^{(2)}\frac{t_2}{l_u}\left(\frac{l}{l_u} - 1\right)\phi(t_1),$$
$$a_{10} = m_4\left(\frac{l}{l_u} - 1\right) + c_{xz}^{(2)}\left(1 - \frac{l}{l_u}\right) + a_3, \qquad a_{11} = c_{xz}^{(2)}\left(\frac{l}{l_u} - 1\right) + m_5,$$
$$a_{12} = m_3\left(\frac{l^3}{l_u^4} - \frac{l}{l_u^2}\right), \qquad a_{13} = m_3m_{16}, \qquad a_{14} = m_3m_{17}, \qquad a_{15} = m_4\frac{m_{18}}{l_u}, \qquad a_{16} = m_4\frac{m_{19}}{l_u},$$

$$a_{17}=m_4\frac{m_{20}}{l_u},\qquad a_{18}=c_{xx}^{(2)}\left(\frac{l^3}{l_u^4}-\frac{l}{l_u^2}\right)t_2,\qquad a_{19}=c_{xx}^{(2)}t_2m_{16},\qquad a_{20}=c_{xx}^{(2)}t_2m_{17},$$

$$a_{21}=c_{xz}^{(2)}\frac{t_1}{l_u}\frac{h+t_1}{2h^2}m_{18},\qquad a_{22}=c_{xz}^{(2)}\frac{t_1}{l_u}\frac{h+t_1}{2h^2}m_{19},\qquad a_{23}=c_{xz}^{(2)}\frac{t_1}{l_u}\frac{h+t_1}{2h^2}m_{20},$$

$$a_{24}=\frac{m_7}{l_u}\left(\frac{l}{l_u}-1\right)+\frac{m_6}{3}\left(\frac{l^3}{l_u^2}-l_u\right)+c_{xx}^{(2)}\left(\frac{l}{l_u^2}-\frac{1}{l_u}\right)t_2\phi^2(t_1),$$

$$a_{25}=m_9\left(\frac{l}{l_u}-1\right)+c_{xz}^{(2)}\left(1-\frac{l}{l_u}\right)\phi(t_1)+m_9-m_5\phi(t_1),\qquad a_{26}=c_{xz}^{(2)}\left(\frac{l}{l_u}-1\right)\phi(t_1)+m_5\phi(t_1),$$

$$a_{27}=m_7\left(\frac{l^3}{l_u^4}-\frac{l}{l_u^2}\right)+m_6m_{18},\qquad a_{28}=m_7m_{16}+m_6m_{19},\qquad a_{29}=m_7m_{17}+m_6m_{20},$$

$$a_{30}=m_9\frac{m_{18}}{l_u}+2m_8m_{25},\qquad a_{31}=m_9\frac{m_{19}}{l_u}+2m_8m_{26},\qquad a_{32}=m_9\frac{m_{20}}{l_u}+2m_8m_{27},$$

$$a_{33}=c_{xx}^{(2)}\left(\frac{l^3}{l_u^4}-\frac{l}{l_u^2}\right)t_2\phi(t_1),\qquad a_{34}=c_{xx}^{(2)}t_2\phi(t_1)m_{16},\qquad a_{35}=c_{xx}^{(2)}t_2\phi(t_1)m_{17},$$

$$a_{36}=c_{xz}^{(2)}\frac{h+t_1}{2h^2l_u}t_1\phi(t_1)m_{18},\qquad a_{37}=c_{xz}^{(2)}\frac{h+t_1}{2h^2l_u}t_1\phi(t_1)m_{19},\qquad a_{38}=c_{xz}^{(2)}\frac{h+t_1}{2h^2l_u}t_1\phi(t_1)m_{20},$$

$$a_{39}=a_{10}-a_3,\qquad a_{40}=a_{25}+a_5,\qquad a_{41}=m_{10}(l-l_u)+m_{11}(l-l_u),\qquad a_{42}=(l_u-l)m_{14},$$

$$a_{43}=m_9\left(\frac{l^3}{l_u^3}-\frac{l}{l_u}\right),\qquad a_{44}=m_9m_{16}l_u,\qquad a_{45}=m_9m_{17}l_u,\qquad a_{46}=m_{10}m_{18},$$

$$a_{47}=m_{10}m_{19},\qquad a_{48}=m_{10}m_{20},\qquad a_{49}=c_{xz}^{(2)}\left(\frac{l}{l_u}-\frac{l^3}{l_u^3}\right)m_{21},\qquad a_{50}=-c_{xz}^{(2)}m_{16}l_um_{21},$$

$$a_{51}=-c_{xz}^{(2)}m_{17}l_um_{21},\qquad a_{52}=-c_{zz}^{(2)}m_{18}m_{22},\qquad a_{53}=-c_{zz}^{(2)}m_{19}m_{22},$$

$$a_{54}=-c_{zz}^{(2)}m_{20}m_{22},\qquad a_{55}=c_{zz}^{(2)}\frac{20t_1h^3+5t_2^2h^2+15t_1t_2^2h+4t_2^4}{5h^2t_2(h+t_1)^2}(l-l_u)+m_{15},$$

$$a_{56}=c_{xz}^{(2)}\left(\frac{l^3}{l_u^3}-\frac{l}{l_u}\right)m_{23},\qquad a_{57}=c_{xz}^{(2)}m_{16}l_um_{23},\qquad a_{58}=c_{xz}^{(2)}m_{17}l_um_{23},\qquad a_{59}=c_{zz}^{(2)}m_{18}m_{24},$$

$$a_{60}=c_{zz}^{(2)}m_{19}m_{24},\qquad a_{61}=c_{zz}^{(2)}m_{20}m_{24},\qquad a_{62}=m_7m_{29}+m_6m_{19},\qquad a_{63}=m_7m_{31}+m_6m_{20},$$

$$a_{64}=m_7m_{32}+m_6m_{33},\qquad a_{65}=m_9m_{26}+2m_8m_{36},\qquad a_{66}=m_9\frac{m_{37}}{l_u^2}+2m_8\frac{m_{37}}{l_u^2},$$

$$a_{67}=m_9\frac{m_{38}}{l_u^2}+2m_8m_{39},\qquad a_{68}=m_7m_{40}+m_6m_{33},\qquad a_{69}=m_7m_{41}+m_6m_{42},$$

$$a_{70}=m_9\frac{m_{43}}{l_u^2}+2m_8m_{44},\qquad a_{71}=m_9\frac{m_{45}}{l_u}+2m_8\frac{m_{38}}{l_u^2},\qquad a_{72}=m_9\frac{m_{46}}{l_u^2}+2m_8m_{47},$$

$$a_{73}=m_7m_{48}+m_6m_{49},\qquad a_{74}=m_9\frac{m_{50}}{l_u^2}+2m_8m_{51},\qquad a_{75}=m_9\frac{m_{52}}{l_u^2}+2m_8\frac{m_{53}}{l_u^2},$$

$$a_{76}=m_9\frac{m_{54}}{l_u}+2m_8m_{55},\qquad a_{77}=m_{10}\frac{m_{57}}{l_u}+4m_{12}m_{59},\qquad a_{78}=m_{10}\frac{m_{60}}{l_u}+4m_{12}m_{61},$$

$$a_{79}=m_{10}\frac{m_{62}}{l_u}+4m_{12}m_{63},\qquad a_{80}=m_{10}\frac{m_{62}}{l_u}+4m_{12}m_{64},\qquad a_{81}=m_{10}\frac{m_{65}}{l_u}+4m_{12}m_{66},$$

$$a_{82}=m_{10}\frac{m_{67}}{l_u}+4m_{12}m_{68},\qquad a_{83}=c_{xx}^{(2)}m_{28}m_{29}+c_{55}^{(2)}m_{19}m_{30},\qquad a_{84}=c_{xx}^{(2)}m_{28}m_{31}+c_{55}^{(2)}m_{20}m_{30},$$

$$a_{85}=c_{xx}^{(2)}m_{28}m_{32}+c_{55}^{(2)}m_{30}m_{33},\qquad a_{86}=c_{xz}^{(2)}m_{26}m_{34}+2c_{55}^{(2)}m_{35}m_{36},$$

$$
\begin{aligned}
&a_{87} = c_{xz}^{(2)} m_{34} \frac{m_{37}}{l_u^2} + 2c_{55}^{(2)} m_{35} \frac{m_{37}}{l_u^2}, \qquad a_{88} = c_{xz}^{(2)} m_{34} \frac{m_{38}}{l_u^2} + 2c_{55}^{(2)} m_{35} m_{39}, \\
&a_{89} = c_{xx}^{(2)} m_{28} m_{40} + c_{55}^{(2)} m_{30} m_{33}, \qquad a_{90} = c_{xx}^{(2)} m_{28} m_{41} + c_{55}^{(2)} m_{30} m_{42}, \\
&a_{91} = c_{xz}^{(2)} m_{34} \frac{m_{43}}{l_u^2} + 2c_{55}^{(2)} m_{35} m_{44}, \qquad a_{92} = c_{xz}^{(2)} m_{34} \frac{m_{45}}{l_u} + 2c_{55}^{(2)} m_{35} \frac{m_{38}}{l_u^2}, \\
&a_{93} = c_{xz}^{(2)} m_{34} \frac{m_{46}}{l_u^2} + 2c_{55}^{(2)} m_{35} m_{47}, \qquad a_{94} = c_{xx}^{(2)} m_{28} m_{48} + c_{55}^{(2)} m_{30} m_{49}, \\
&a_{95} = c_{xz}^{(2)} m_{34} \frac{m_{50}}{l_u^2} + 2c_{55}^{(2)} m_{35} m_{51}, \qquad a_{96} = c_{xz}^{(2)} m_{34} \frac{m_{52}}{l_u^2} + 2c_{55}^{(2)} m_{35} \frac{m_{53}}{l_u^2}, \\
&a_{97} = c_{xz}^{(2)} m_{34} \frac{m_{54}}{l_u} + 2c_{55}^{(2)} m_{35} m_{55}, \qquad a_{98} = c_{xz}^{(2)} m_{56} \frac{m_{57}}{l_u t_2} + c_{55}^{(2)} m_{58} m_{59}, \\
&a_{99} = c_{xz}^{(2)} m_{56} \frac{m_{60}}{l_u t_2} + c_{55}^{(2)} m_{58} m_{61}, \qquad a_{100} = c_{xz}^{(2)} m_{56} \frac{m_{62}}{l_u t_2} + c_{55}^{(2)} m_{58} \frac{m_{63}}{4}, \\
&a_{101} = c_{xz}^{(2)} m_{56} \frac{m_{62}}{l_u t_2} + c_{55}^{(2)} m_{58} m_{64}, \qquad a_{102} = c_{xz}^{(2)} m_{56} \frac{m_{65}}{l_u t_2} + c_{55}^{(2)} m_{58} m_{66}, \\
&a_{103} = c_{xz}^{(2)} m_{56} \frac{m_{67}}{l_u t_2} + c_{55}^{(2)} m_{58} m_{68}, \qquad a_{104} = a_2 - a_1 \phi(t_1), \\
&a_{105} = a_9 - a_8 \phi(t_1), \qquad a_{106} = a_4 - a_2 \phi(t_1), \qquad a_{107} = a_{24} - a_9 \phi(t_1), \\
&a_{108} = a_{25} - a_{10} \phi(t_1), \qquad a_{109} = a_{26} - a_{11} \phi(t_1), \qquad a_{110} = a_{27} - a_{12} \phi(t_1), \\
&a_{111} = a_{28} - a_{13} \phi(t_1), \qquad a_{112} = a_{29} - a_{14} \phi(t_1), \qquad a_{113} = a_{30} - a_{15} \phi(t_1), \\
&a_{114} = a_{31} - a_{16} \phi(t_1), \qquad a_{115} = a_{32} - a_{17} \phi(t_1), \qquad a_{116} = a_{33} - a_{18} \phi(t_1), \\
&a_{117} = a_{34} - a_{19} \phi(t_1), \qquad a_{118} = a_{35} - a_{20} \phi(t_1), \qquad a_{119} = a_{36} - a_{21} \phi(t_1), \\
&a_{120} = a_{37} - a_{22} \phi(t_1), \qquad a_{121} = a_{38} - a_{23} \phi(t_1).
\end{aligned} \tag{A.3}
$$

$$
\begin{aligned}
b_0 &= c_{55}^{(1)} \int_0^{l_u} f_2^2(x)\,\mathrm{d}x + \frac{c_{55}^{(1)}}{3} \left(\frac{l^3}{l_u^2} - l_u \right) f_2^2(l_u) + 2c_{55}^{(1)} m_{18} f_2(l_u) w_3 + 2c_{55}^{(1)} m_{19} f_2(l_u) w_4 \\
&\quad + 2c_{55}^{(1)} m_{20} f_2(l_u) w_5 + c_{55}^{(1)} m_{19} w_3^2 + 2c_{55}^{(1)} m_{20} w_3 w_4 + 2c_{55}^{(1)} m_{33} w_3 w_5 + c_{55}^{(1)} m_{33} w_4^2 \\
&\quad + 2c_{55}^{(1)} m_{42} w_4 w_5 + c_{55}^{(1)} m_{49} w_5^2, \\
b_1 &= c_{xx}^{(1)} \int_0^{l_u} \left[f_2'(x) \right]^2 \mathrm{d}x + c_{xx}^{(1)} \left(\frac{l}{l_u^2} - \frac{1}{l_u} \right) f_2^2(l_u) + 2c_{xx}^{(1)} \left(\frac{l^3}{l_u^4} - \frac{l}{l_u^2} \right) f_2(l_u) w_3 + 2c_{xx}^{(1)} m_{16} f_2(l_u) w_4 \\
&\quad + 2c_{xx}^{(1)} m_{17} f_2(l_u) w_5 + c_{xx}^{(1)} m_{29} w_3^2 + 2c_{xx}^{(1)} m_{31} w_3 w_4 + 2c_{xx}^{(1)} m_{32} w_3 w_5 + c_{xx}^{(1)} m_{40} w_4^2 \\
&\quad + 2c_{xx}^{(1)} m_{41} w_4 w_5 + c_{xx}^{(1)} m_{48} w_5^2, \\
b_2 &= c_{xz}^{(1)} \frac{3}{t_1^3} \int_0^{l_u} f_2'(x) f_3(x)\,\mathrm{d}x + c_{xz}^{(1)} \frac{3}{t_1^3} \left(\frac{l}{l_u} - 1 \right) f_2(l_u) f_3(l_u) + c_{xz}^{(1)} \frac{3}{t_1^3} \frac{m_{18}}{l_u} f_2(l_u) w_6 \\
&\quad + c_{xz}^{(1)} \frac{3}{t_1^3} \frac{m_{19}}{l_u} f_2(l_u) w_7 + c_{xz}^{(1)} \frac{3}{t_1^3} \frac{m_{20}}{l_u} f_2(l_u) w_8 + c_{xz}^{(1)} \frac{3}{t_1^3} \left(\frac{l^3}{l_u^3} - \frac{l}{l_u} \right) f_3(l_u) w_3 \\
&\quad + c_{xz}^{(1)} \frac{3}{t_1^3} l_u m_{16} f_3(l_u) w_4 + c_{xz}^{(1)} \frac{3}{t_1^3} l_u m_{17} f_3(l_u) w_5 + c_{xz}^{(1)} \frac{3}{t_1^3} m_{26} w_3 w_6 + c_{xz}^{(1)} \frac{3}{t_1^3} \frac{m_{37}}{l_u^2} w_3 w_7 \\
&\quad + c_{xz}^{(1)} \frac{3}{t_1^3} \frac{m_{38}}{l_u^2} w_3 w_8 + c_{xz}^{(1)} \frac{3}{t_1^3} \frac{m_{43}}{l_u^2} w_4 w_6 + c_{xz}^{(1)} \frac{3}{t_1^3} \frac{m_{45}}{l_u} w_4 w_7 + c_{xz}^{(1)} \frac{3}{t_1^3} \frac{m_{46}}{l_u^2} w_4 w_8 \\
&\quad + c_{xz}^{(1)} \frac{3}{t_1^3} \frac{m_{50}}{l_u^2} w_5 w_6 + c_{xz}^{(1)} \frac{3}{t_1^3} \frac{m_{52}}{l_u^2} w_5 w_7 + c_{xz}^{(1)} \frac{3}{t_1^3} \frac{m_{54}}{l_u} w_5 w_8 \\
&\quad - c_{55}^{(1)} \frac{3}{t_1^3} \int_0^{l_u} f_2(x) f_3'(x)\,\mathrm{d}x - c_{55}^{(1)} \frac{6}{t_1^3} m_{25} f_2(l_u) w_6 - c_{55}^{(1)} \frac{6}{t_1^3} m_{26} f_2(l_u) w_7 - c_{55}^{(1)} \frac{6}{t_1^3} m_{27} f_2(l_u) w_8
\end{aligned}
$$

$$
\begin{aligned}
&- c_{55}^{(1)} \frac{6}{t_1^3} m_{36} w_3 w_6 - c_{55}^{(1)} \frac{6}{t_1^3} \frac{m_{37}}{l_u^2} w_3 w_7 - c_{55}^{(1)} \frac{6}{t_1^3} m_{39} w_3 w_8 - c_{55}^{(1)} \frac{6}{t_1^3} m_{44} w_4 w_6 - c_{55}^{(1)} \frac{6}{t_1^3} \frac{m_{38}}{l_u^2} w_4 w_7 \\
&- c_{55}^{(1)} \frac{6}{t_1^3} m_{47} w_4 w_8 - c_{55}^{(1)} \frac{6}{t_1^3} m_{51} w_5 w_6 - c_{55}^{(1)} \frac{6}{t_1^3} \frac{m_{53}}{l_u^2} w_5 w_7 - c_{55}^{(1)} \frac{6}{t_1^3} m_{55} w_5 w_8, \\
b_3 &= c_{xx}^{(1)} \int_0^{l_u} f_1'(x) f_2'(x) dx + c_{xx}^{(1)} \left(\frac{l}{l_u^2} - \frac{1}{l_u} \right) f_1(l_u) f_2(l_u) + c_{xx}^{(1)} \left(\frac{l^3}{l_u^4} - \frac{l}{l_u^2} \right) f_1(l_u) w_3 \\
&+ c_{xx}^{(1)} m_{16} f_1(l_u) w_4 + c_{xx}^{(1)} m_{17} f_1(l_u) w_5.
\end{aligned} \tag{A.4}
$$

$$
\begin{aligned}
k_0 &= a_7(a_1 a_4 - a_2^2), \qquad k_1 = a_1 a_5^2 - a_1 m_6 a_7 - a_1 a_4 a_6 + 2 a_2 a_3 a_5 + a_2^2 a_6 + a_3^2 a_4, \\
k_2 &= m_6(a_1 a_6 - a_3^2), \qquad k_3 = a_1 a_5 + a_2 a_3, \qquad k_4 = a_1 m_6, \qquad k_5 = a_2^2 - a_1 a_4, \\
k_6 &= \frac{1}{a_3} \left\{ \frac{k_4 + k_5(\alpha^2 - \beta^2)}{k_3} \left[\frac{a_6(\alpha^2 - \beta^2)}{(\alpha^2 + \beta^2)^2} - a_7 \right] + \frac{4 k_5 \alpha^2 \beta^2 a_6}{k_3 (\alpha^2 + \beta^2)^2} + a_5 \right\}, \\
k_7 &= \frac{1}{a_3} \left\{ \frac{2 k_5 \alpha \beta}{k_3} \left[\frac{a_6(\alpha^2 - \beta^2)}{(\alpha^2 + \beta^2)^2} - a_7 \right] - \frac{2 \alpha \beta a_6}{(\alpha^2 + \beta^2)^2} \frac{k_4 + k_5(\alpha^2 - \beta^2)}{k_3} \right\}, \\
k_8 &= \frac{k_4 + k_5(\alpha^2 - \beta^2) + 2 k_5 \beta^2}{k_3(\alpha^2 + \beta^2)} \alpha, \qquad k_9 = \frac{2 k_5 \alpha^2 - k_4 - k_5(\alpha^2 - \beta^2)}{k_3(\alpha^2 + \beta^2)} \beta, \qquad k_{10} = \sinh(\alpha l_u), \\
k_{11} &= \sinh(\beta l_u), \qquad k_{12} = \cosh(\alpha l_u), \qquad k_{13} = \cosh(\beta l_u), \qquad k_{14} = \cos(\beta l_u), \qquad k_{15} = \sin(\beta l_u), \\
k_{16} &= \cos(\alpha l_u), \qquad k_{17} = \sin(\alpha l_u), \qquad k_{18} = \sinh(\alpha l_u) \sin(\beta l_u), \qquad k_{19} = \cosh(\alpha l_u) \cos(\beta l_u), \\
k_{20} &= \sinh(\alpha l_u) \cos(\beta l_u), \qquad k_{21} = \cosh(\alpha l_u) \sin(\beta l_u).
\end{aligned} \tag{A.5}
$$

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